DETERMINISTIC MODEL STUDY OF RIVER FORMATION

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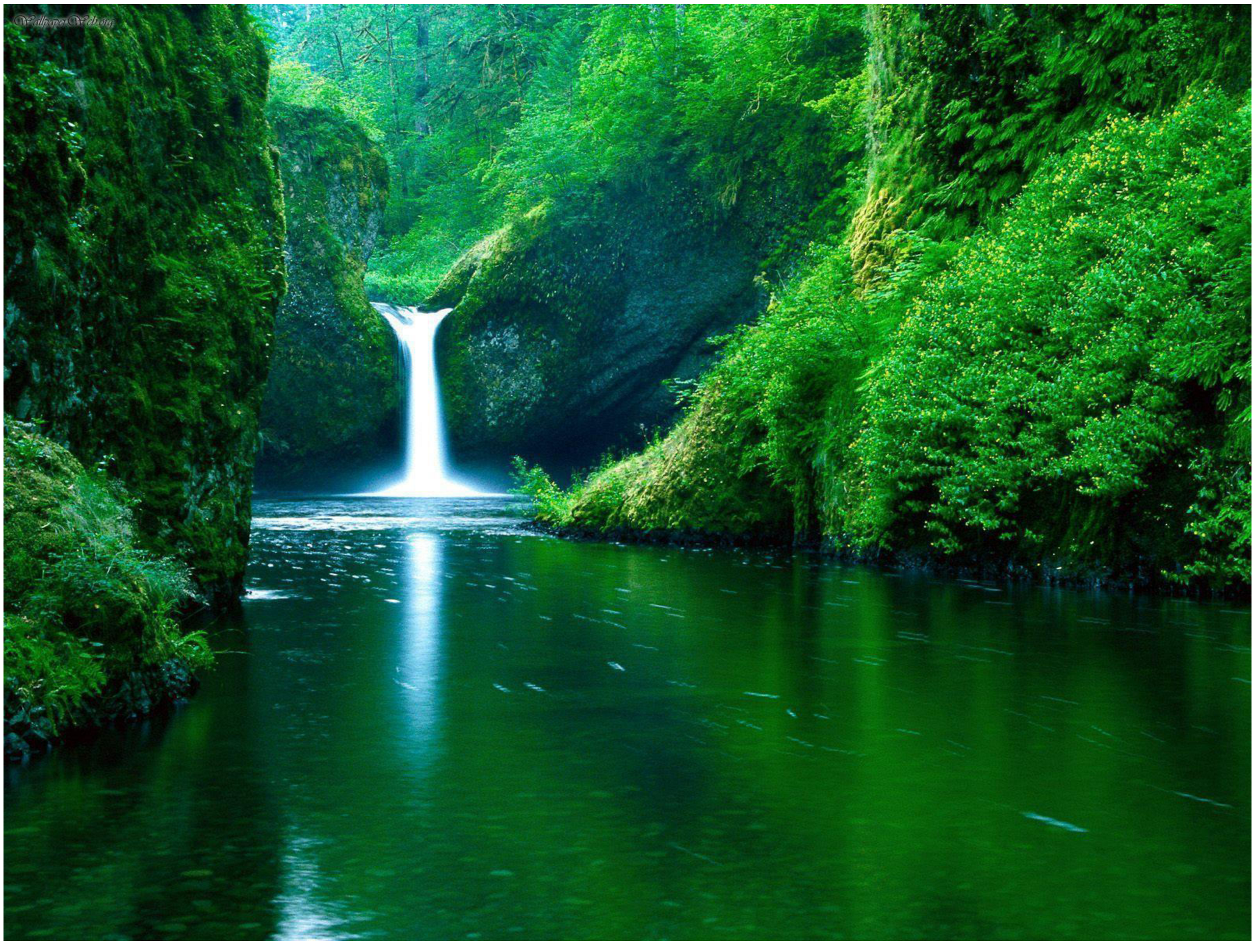

Eye catching river flow of Bain Ganga (Pachmarhis, India)

Abstract:- *In this paper, it is considered that mathematical and numerical analysis of a deterministic model describing river channel formation and the evolution of its depth. The model involves a degenerate nonlinear parabolic equation (satisfied on the interior of the support of the solution) with a super-linear source term and a prescribed constant mass. It is proposed here that a formulation of the problem which allows us to show the existence of a solution and leads to a suitable numerical scheme for its approximation. A particular novelty of the model is that the evolving channel self-determines its own width, without the need to pose any extra conditions at the channel margin. The theme of the article is a free boundary formulation for the river formation giving rise to a measure on the free boundary.*

Keywords: *River Formation, Free Boundary Problems, Measure on the Free Boundary.*

Introduction

The main results of the work is presented here, concerning the deterministic model for the river channel formation introduced by A.C. Fowler, N. Kopteva and C. Oakley (2007), Diaz, J.I. Fowler, A. C. Mufioz, A. I and Schiavi, E. (2008) The ingredients of a model are variables describing channel flow and sediment transport, and the mechanism of channel formation arises through an instability, in which locally increased flow causes increased erosion, which in turn increases the flow depth and thus also the flow. This positive feedback induces instability, as was shown by Smith and Bretherton (1972). The starting point was a coupled set of partial differential equations describing $s(x,y,t)$, the hillslope elevation, and $h(x,y,t)$, the water depth

$$\nabla.(h\mathbf{u}) = r, s_t + \nabla.\mathbf{q} = U, \tag{1.1}$$

this represents conservation of mass of flow and sediment. The mean flow velocity u is determined through a momentum balance equation, while the sediment flux $\mathbf{q}$ is usually taken as an empirically required function of flow-induced bed stress and bed slope, the resulting combination (the effective bed stress) being denoted τ. The source term r represents rainfall, while U represents tectonic uplift. The time derivative in the water mass equation is ignored. It is assumed that it has been written in dimensionless form, so that the variables are $O(1)$. One can show that suitable models for the flow speed u and effective bed stress τ are

$$\mathbf{u} = \mathrm{h}^{1/2}|\nabla\eta|^{1/2}\mathbf{n}, \boldsymbol{\tau} = \mathbf{u}|\mathbf{u}| - \beta\nabla_{\mathbf{S},} \tag{1.2}$$

where typically $\beta = O(1)$, and the down-water slope normal n is defined by $\mathrm{n} = -\frac{\nabla\eta}{|\nabla\eta|}$, η represents the water surface elevation, and in dimensionless terms is related to hill slope elevation s and water film thickness h by $\eta = s+\delta h$. The parameter δ is very small, a typical estimate being 10^{-5}. Finally, the sediment flux is taken to have the form $\mathbf{q} = V(\tau)N$,where $\tau = |\tau|$ and the down-sediment flow normal N is $\mathrm{N} = \frac{\tau}{\tau}$. V is an increasing function of τ, with $V \approx \tau^{3/2}$ being a popular choice (this essentially stemming from the model of Meyer-Peter and Muller, 1948). Uniform overland flow is unstable to y dependent perturbations of small wavelength, and it can be examined the nonlinear evolution of these by directly seeking asymptotic expansions in terms of δ. To do so, it is firstly supposed that the channels which form are aligned in the X direction, and (sensibly) that the perturbation to the water surface is small, comparable to the overland flow depth: $\eta = \eta_0 + \delta Z$. It may be then linearize the geometry of the system, to find that $\mathbf{n} = \mathbf{i} - \frac{\delta Z_{\mathcal{Y}}}{S}\mathrm{j} + \cdots$, and $\mathrm{N} = \mathrm{i} - \frac{\delta}{S}\left\{Z_{\mathcal{Y}} - \frac{\beta}{h+\beta}h_{\mathcal{Y}}\right\}\mathrm{j}+\ldots$, where j is the unit vector in the $\mathcal{Y}$direction and $S(X) = |\eta_0'(X)|$ is the unperturbed downhill slope. The nonlinear channel evolution then arises from a rescaling of the hill slope evolution equation, in which it is put $\mathcal{Y} = \delta^{1/2}Y, h = \frac{H}{\delta^{1/3,}}$ $t = \delta^{7/6}T$; after some algebra, it is found that the leading order sediment transport equation takes the form $\frac{\partial H}{\partial H} = s'^{S^{1/2}H^{3/2}} + S^{1/2}\frac{\partial}{\partial Y}\left[\beta H^{\frac{1}{2}}\frac{\partial H}{\partial Y}\right]$, where $S' = ds/dX$.

It is important to realize that this equation arises through conservation of sediment. Only Y derivatives are present, because the lateral length scale is so much smaller than the downslope one. The perturbation Z to the water surface is in fact then determined by quadrature of the water conservation equation, but integration of this equation in the across stream direction yields the integral constraint$\int_{-\infty}^{\infty} H^{3/2}dY = \frac{2LrX}{S^{1/2}}$, , where L is the spacing (on the original hill slope length scale for $\mathcal{Y}$) between channels; the limits in the integral are, however, infinite because the integral is with respect to the much smaller channel width length scale. Suitable initial and boundary conditions for the channel depth are that $H \to 0$ as $Y \to \pm\infty, H = H_0(Y)$ at $T = 0$. the above equation, together with the integral constraint and initial/boundary conditions, forms the basis of our study. It will be assumed that $S' > 0$, so that the nonlinear term in the H equation is a source. We define $H = \left(\frac{6}{\beta}\right)^{1/3}(LrX)^{2/3}u$, $T = \left(\frac{\beta}{6}\right)^{1/6}\frac{t}{S^{1/2}S'(LrX)^{1/3'}}$ $Y = \left(\frac{2\beta}{3S'}\right)^{1/2}x$.

2 Mathematical analysis

It is considered that the problem which is assuming an initial thickness perturbation $u_0(x)$ satisfying natural physically based hypothesis, i.e., a bounded and non-negative function with a compact and connected support $[-\zeta_{0,}\zeta 0]$ such that form $\int_0^{+\infty} u_0^m(x)\,dx = M/2$, for $m>1$ (so including the case of $m = 3/2$ as before. For the sake of simplicity of the exposition it is also assumed symmetric initial data. It will be especially interested in the question of *global solvability* (in time) of the following problem: find a continuous curve

$\zeta: [0,+\infty) \to IR^{+}$and a function $u: P \longrightarrow [0,+\infty)$(regular enouh)such that\

$$
(SL)\begin{cases}
u_t = (u^m)_{xx} + u^m, & \text{in } D'(P), \\
u(x,0) = u_0(x) & \text{a. e } x\epsilon\, \Omega_{0,} \\
u(x,t) > 0, \ \text{a. e.}\,(x,t)\epsilon\, P_{,} \text{ and } u(x,t) \equiv 0, & \text{a. e } (x,t) \notin P \\
u(\zeta(t),t) = 0, \ (u^m)_x\,(0,t) = 0 & \text{a. e. } t \in (0,+\infty), \\
\zeta(0) = \zeta_0 \ \text{ and } \zeta(t) > 0 \displaystyle\int_0^{+\infty} u^m(x,t,)dx = \frac{M}{2} & \text{a. e. } t\,\epsilon(0,+\infty).
\end{cases}
$$

where $\Omega_0 = (0,\zeta_0), \Omega_t = (0,\zeta(t)) \times \{t\}, P = U_{t>0}\Omega_t$. Notice that $D'(P)$ denotes the space of *distributions* on P and is P the *positivity subset* of the solution. Later on it can be made more precise the (minimal) regularity of the sol P solution. The function $\zeta(t)$ is called *the interface* separating the (connected) region where $u(x,t) > 0$ from the region where $u(x,t) = 0$. it is unknown and it is usually called the *free* or *moving boundary* of the problem. Due to the free boundary it is referred to the strong formulation (SL) as the *strong-local* formulation. It is emphasized that the mass conservation constraint given in (SL) prevents possible blow-up phenomena which could arise (without this condition) due to the presence of the source term u^min the equation.

An important difficulty, in order to get a global formulation (i.e. extended to the whole domain $(x,t)\epsilon(0,+\infty) \times (0,+\infty)$, and not only on $(x,t)\epsilon\, P)$, is the necessity to provide a suitable description of the flux $-(u^m)_x(\zeta(t)t,)$at the free boundary. This leads to a new constrained global formulation suitable for mathematical analysis and numerical resolution. Problems of this type arise in fluid mechanics (problems of the Bernoulli type), in combustion and in plasma physics [Diaz, J.I., Padial J. F. and Rakotoson, J.M. (2007)]

To prove the existence, an auxiliary global formulation can be used on the whole domain $IR^{+} \times [0,T]$. to be precise it is introduced that the notation $\delta_{\partial\{u(t,\cdot)=0\}}$to design the Dirac delta distribution located at the interface $x = \zeta(t)$ for each $t\,\epsilon(0,T)$ $(i.e.\,\delta_{\partial\{u(t,\cdot)=0\}}) = \delta_{(\zeta(t),t)}$.

The reformulation of the mass constraint requires the "zero total measure" condition. So, the global formulation is:

$$
(P)\begin{cases}
u_t = (u^m)_{xx} + u^m, -\dfrac{M}{2}\delta_{\partial\{u(t,\cdot)=0\},} & D'(IR^{+} \times (0,T)), \\
u(x,0) = u_0(x) & \text{a. e. } x \in (0,+\infty), \\
u_x(0,t) = 0, u(x,t) \to 0 \text{ as } x \to +\infty & \text{a. e. } t \in (0,T), \\
\mu\ (t,\cdot) = u_t(t,\cdot) - (u^m)_{xx}(t,\cdot) \text{ and } \displaystyle\int_0^{+\infty} d\mu(t,\cdot) = 0, \text{a. e. } t \in (0,T),
\end{cases} \quad (2.3)
$$

It is used that a two steps iterative approximation. The main idea is to construct the sequence $\{u_{2n+1}: n = 0,1,2 \ldots\}$ as solutions of problems

$$(P_{2n+1})\begin{cases}(u_{2n+1})t = ((u_{2n+1})^m)_{xx} + (u_{2n})^{m-1}(u_{2n+1}) - \frac{M}{2}\delta_{\partial\{u_{2n+1}(t,\cdot)=0\}}, D'(IR^+ \times (0,T)), \\ (u_{2n+1})(x,0) = u_0(x) \quad \text{a.e.} x \in (0,+\infty), \\ (u_{2n+1})x\,(0,t) = 0, (u_{2n+1})\,(x,t) \to 0 \text{ as } x \to +\infty \quad \text{a.e.} t \in (0,T),\end{cases}$$

(where for $n = 0$ it is used as u_{2n} the initial condition u_0) and then $\{u_{2n} : n = 1,2 \ldots.\}$ By

$$(P_{2n})\begin{cases}u_{2n}(x,t) = C_{2n}(t)u_{2n-1}(x,t) \text{ for a.e. } (x,t) \in IR^+ \times (0,T), \\ \int_0^{+\infty} (u_{2n}(x,t))^{m-1}(u_{2n-1}(x,t))dx = \frac{M}{2} \text{ for a.e.} t \in (0,T),\end{cases}$$

for some $C_{2n}(t) > 0$. For the detailed proof of the convergence of the algorithm see Diaz et al.

Theorem *there exists a function* $C^*(t) > 0, C^* \in L^\infty(0,T)$ *and a function* $u \in C([0,T] : L^1(IR^+))$ *Such that*

$$\begin{cases}u_t = (u^m)_{xx} + C^*(t)^{m-1}u^m - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, \quad D'(IR^+ \times (0,T)), \\ u(x,0) = u_0(x) \quad \text{a.e.} x \in (0,+\infty), \\ u_x(0,t) = 0, u(x,t) \to 0 \text{ as } x \to +\infty \quad \text{a.e.} t \in (0,T),\end{cases}$$

and $C^*(t)^{m-1}\int_0^{+\infty} u(x,t)^m dx = \frac{M}{2}$.

Concerning the numerical resolution of the problem *(P)*, for each initial condition h_0, its mass is computed, say *M/2*, and the associated stationary solution $v(x)$to which the solution should converge when $t \to +\infty$, see Fowler et al., 2007. In order to discretize with respect to the coordinate *x*, at each time level $l.dt$, piecewise linear finite elements will be employed $L_{l,k} := \{\phi \in C^0([0,+\infty)) : \phi|E \in \mathbf{P}_1, \forall E \in \mathbf{T}_{l,k}\}$ in a uniform grid, $\mathbf{T}_{l,k}$,of step k Also, $\mathbf{B}_{lk} := \{\phi_i\}$ is a base of finite linear elements in $L_{l,k}$. Then, the discretized problem is formulated as follows: *Find* $(u_{l+1})_k \in L_{l,k}$ $(u_l + 1)_k = \sum_j (u_{l+1})_k^j \phi_j$, *such that*

$$\int_{T_{l,k}} (u_{l+1})_k \phi_i dx = \int_{T_{l,k}} (u_l)_k \phi_i dx - \frac{3dt}{2}\int_{T_{l,k}} (u_{n+1})_k^{\frac{1}{2}}((u_{l+1})_k)_x \phi_{i_x} dx$$
$$+dt\int_{T_{l,k}} (u_{l+1})_k^{\frac{3}{2}} \phi_i dx - dt\int_{T_{l,k}} \frac{M}{2}\delta(u_l)\phi_i dx, \forall\phi_i \in \mathbf{B}_{1,k}. \quad (2.4)$$

In order to deal with the nonlinearities, It is considered that the iterative scheme: for p=2n+1 from 1 to N, *n* = 0,1, 2..., and *N* an odd number to be fixed, to consider the problem,

$$\int_{T_{l,k}} (u_{l+1,2n+1})_k \phi_i\, dx = \int_{T_{l,k}} (u_l)_k \phi_i dx - \frac{3dt}{2}\int_{T_{l,k}} (u_{l+1,2n})_k^{\frac{1}{2}}\left((u_{l+1,2n+1})_k\right)_x \phi_{ix} dx$$
$$+dt\int_{T_{l,k}} (u_{l+1,2n})_k^{\frac{1}{2}} (u_{l+1,2n+1})_k \phi_i dx - dt\int_{T_{l,k}} \frac{M}{2}\delta(u_l)\phi_i dx, \quad \forall\phi_i \in \mathbf{B}_{1,k}. \quad (2.5)$$

where, $(u_{l+1,2n})_k$ has been rescaled before being introduce in (2.5) so that $\int (u_{l+1,2n})_k^{\frac{3}{2}} = \frac{M}{2}$, according to $(P_{2n}), i.e, (u_{l+1,2n})_k = C_{l+1,2n}(u_{l+1,2n-1})_k$ ·The resulting system of equations for the nodal values at the $(2n+1)$th −step is solved with the Gauss Seidel method. In order to initiate the iterative scheme, one can take as $\left(u_{l+1,p=1}\right)_k$ the values obtained in the previous time step, that is to say, $\left(u_{l+1,p=1}\right)_k = u_l$. The scheme finishes assuming the values for the $(l+1)$ −time level given by $u_{l+1} = \left(u_{l+1,p=N}\right)_k$.

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