MORPHOMETRIC ANALYSIS OF RIVER BRANCHING SYSTEM

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Abstract

In the river networks, branching systems are a basic geomorphic component. Study of drainage patterns rests largely on junction angles (Zernitz 1932). Inspite of early acceptance of regularities in river branching angles (Playfair 1802) and the importance of angles in understanding the situation of drainage networks, very little theoretical work has been done on this topic. Two theoretical models stand out, however: the Hortonian model (Horton 1926, 1932) and the minimum power model (Howard 1971). Thus far, much of the attention has been focused on the application of the Hortonian

model (Schumm 1956; Lubowe 1964; Howard 1971; Pieri 1979). Because of its simplicity, this model is much more appealing than Howards model. Yet, apparently derived from distinct assumptions, both models yield surprisingly similar theoretical angles for a set of parameters typical of natural streams (Howard 1971).

Angular geometry models are based upon the optimality principle which states that a system will perform its task with utmost efficiency (Rosen 1967). It is then postulated that the geometric configuration of rivers will be such that the costs involved in the operation and/or maintenance of the system will be minimized. This teleological argument is the basis for the derivation of angular geometry models. As I will see, Hortons model can be viewed as one special application of the minimum power losses model. Since several optimality criteria other than minimum power losses can also be envisaged, it is possible to generalize the optimal angular geometry models. Such criteria can be expressed in terms of and simplified by the hydraulic geometry relationships. As a result, a general optimal angular geometry model will be introduced in the second part of the paper. According to the model, optimal angles are a function of the relative size of the tributary streams merging at a junction point and of the hydraulic geometry exponents. A discussion of the model's implications will follow its derivation.

General theoretical framework of Horton s and Howard's models of river branching

Horton (1926, 1932) derived his model from the assumption that overland flow on the valley slopes follows the line of steepest gradient. The angle (θ) formed between the line of overland flow and the stream channel collecting the flow downslope was derived from a trigonometric model and Horton (1932) showed that

$$\cos\theta = S_c/S_g, \tag{1}$$

where $S_c$and S_g are the channel gradient and ground slope respectively (Fig. 1). Hence θ become wider as the ground slope becomes much steeper than the channel gradient. Horton (1945) later adapted the model to the case where one major stream is joined by a tributary stream. The angle of entry then is given by

$$\cos\theta = S_0/S_1, \tag{2}$$

where S_0 and S_1 denote the channel gradients of the receiving and the tributary streams, respectively. Small tributaries developed on steeper slopes should enter a low-gradient large receiving stream at a wide angle (close to 90°).

Horton did not rely on an optimization method. Without knowing, he found the solution to minimum power losses per unit discharge problem, in this case are given by

$$C = \omega_0 l_0 + \omega_1 l_1\,, \tag{3}$$

Where ω_i is the cost per unit length for branch i and l_i is the length of branch i (Fig. 2).

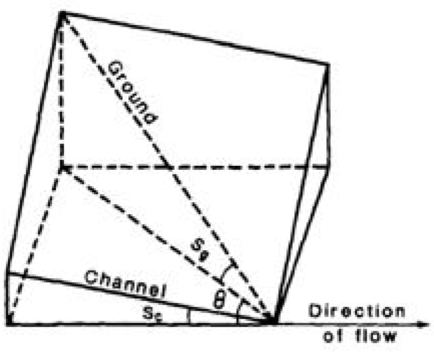

Figure. 1 The Hortonian Model of River Branching Angles (after Horton 1932)

Using an analogy to the law of refraction I can show that total costs will be at a minimum when

$$\cos\theta = \omega_0/\omega_1. \quad (4)$$

(For a complete derivation of this equation, the reader is referred to Rosen (1967) or Roy (1982).) For equation (4) to be applicable, however, the costs must be defined per unit volume per unit length. This is required by the fact that costs also increase with branch size thus implying ω_0 would be larger than to,, which is not consistent with (4).

In a river channel, power losses may be used to estimate the costs involved in the operation of the system. Power losses $(P)$are given by

$$P = \rho g Q S l, \quad (5)$$

where ρ is fluid density; g, gravitational constant; Q,water discharge; S,channel gradient (slope of energy grade line); and l, channel length (Leopold et al. 1964). The cost (ω) per unit length per unit discharge

$$\omega = \rho g S \quad (6)$$

can be substituted into equation (4) to obtain

$$\cos\theta = \rho g S_0/\rho g S_1 \quad (7)$$

Taking into account, fluid density is constant, the Hortonian model is obtained by simplifying equation (7). Thus, Horton (1932) derived an optimal angular geometry model that minimizes the sum of channel lengths weighted by the slope.

Howard (1971) pointed out several shortcomings associated with the Hortonian model. One problem deals with the lack of modification of the course of the receiving stream as it is joined by a tributary stream. In natural systems, however, the main stream is usually deflected at a junction. A second problem arises when streams of nearly equal gradient merge. In that case, the angle of entry should tend towards 0°. Empirical findings (Lubowe 1964) suggest that angles of junction between tributary streams of equal gradient are quite large.

In view of these problems, Howard (1971) defined two angles of entry θ_1 and θ_2 with respect to the axial prolongation of the receiving stream (Fig. 3). Hence, a general Hortonian model was proposed where

$$\cos\theta_1 = S_0/S_1;\ \cos\theta_2 = S_0/S_2. \quad (8)$$

In this Paper, Horton's argument is applicable separately for each side of the junction of the channel. To generalize the model even further, Howard (1971) used the hydraulic geometry relationships in conjunction with the continuity equation

$$Q_0 = Q_1 + Q_2. \quad (9)$$

Considering that, at a junction, the discharge of the main stream is equal to the sum of the discharges of the tributaries, and that the relationship between slope and discharge is given by

$$S = tQ^z; z < 0, \quad (10)$$

Then

$$\cos\theta_1 = \left((Q_1 + Q_2)/Q_1\right)^z; \qquad \cos\theta_1 = \left((Q_1 + Q_2)/Q_1\right)^z;$$

(11)

These equation can be defining a symmetry ratio as

$$\alpha = Q_2/Q_1;\ Q_2 \leqq Q_1, \quad (12)$$

I have

$$Q_2 = \alpha Q_1 \quad (13)$$

and

$$Q_0 = (1+\alpha)Q_1. \quad (14)$$

Equations (13) and (14) can be substituted into (11) to obtain

$$\cos \theta_1 = (1+\alpha)^z;\ \cos \theta_2 = (1/\alpha + 1)^z. \quad (15)$$

So that, the Hortonian model states that optimal angles are a function of the relative size of the tributaries and of the rate of change in slope with discharge. The last step of this analysis although implicit in Howard's argument was never before mathematically formalized.

Howard (1971) not only modified the Hortonian model but also proposed an alternative way to look at the angular geometry problem. He derived a minimum power losses model to predict the angles of entry. Assuming relatively straight channels, the angles of entry are determined by locating the junction point where two tributary streams merge to form one receiving stream (Fig. 2). Given that two tributary streams and one receiving stream, respectively, pass through the points (X_1, Y_1), (X_2, Y_2), and (X_0, Y_0) (see Fig. 3), and given that the costs of transporting the fluid and of maintaining the channel are proportional to channel length (l), it is possible to specify the location of the junction point so as to minimize

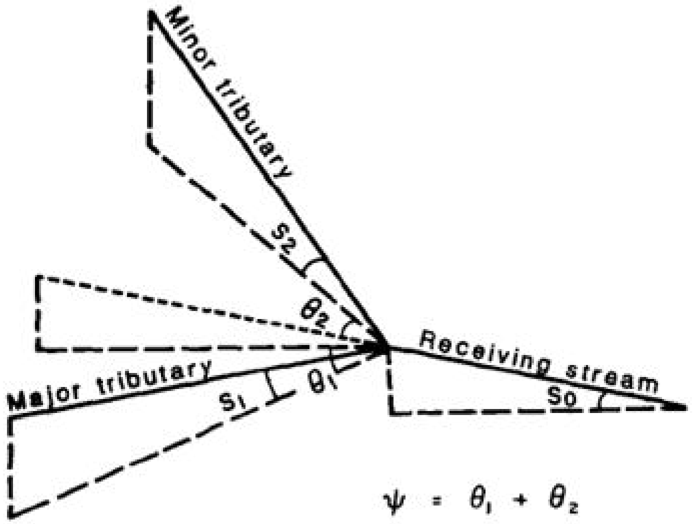

Figure. 2 . Angular Geometry of a River Junction Based on the Concept of Angles of Entry (after Howard 1971)

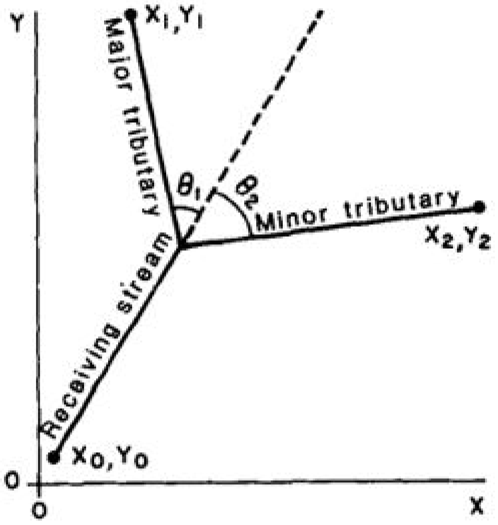

Figure. 3. The Optimal Angular Geometry of a River Junction as a Locational Problem

$$Z = \sum_{i=0}^{2} w_i l_{i,} \quad (16)$$

where ω_i is the cost per unit length associated with branch i. In Euclidean space, length of branch i is

$$l_i = ((X_i - X)^2 + (Y_i - Y)^2)^{1/2}, \quad (17)$$

where X and $Y$are the coordinates of the junction point. Hence, angular geometry of a junction is given by the minimization of the sum of weighted channel lengths.

This problem is identical to the three-point weber problem in economic geography and widely known in location theory (weber 1929; wesolowsky 1973). In such applications, the location in space of the facility (junction point) is the important result of the analysis. In geomorphology , the location of the junction point is important only because it determines the angular geometry of a junction. The solution of the three-point problem is Ill known, and it represents the first step in the derivation of the optimal angular geometry models. As indicated by Howard (1971), this derivation simply requires a trigonometric transformation of the solution for X and Y(For a complete derivation the reader is referred to (Zamir 1976).) The general angular geometry model (Zamir 1976) is given by

$$\cos \theta_1 = (\omega_0^2 + \omega_1^2 - \omega_2^2)/2\omega_0\omega_1 \quad (18)$$

$$\cos \theta_2 = (\omega_0^2 + \omega_2^2 - \omega_1^2)/2\omega_0\omega_2 \quad (19)$$

$$\cos \psi = (\omega_0^2 + \omega_1^2 - \omega_2^2)/2\omega_1\omega_2, \quad (20)$$

where ψ is the junction angle equal to the sum of the angles of entry.

Importantly, particular solutions to the angular geometry problem preceded the introduction of this general model. For arterial branching,to minimize total power losses

$$\omega_i = cr_i^2, \quad (21)$$

where r_i is the radius of the stram and c is a constant. This route was followed in geomorphology, and Howard (1971) also derived a model based on power losses as the optimality criterion. His model results from the substitution of power losses per unit length given by equation (6) into equations (18) to (20).

A Generalization of the Model

Howard's contribution, although of great theoretical importance, represents only one particular application of the optimality principle to river branching azi-gles. Indeed, given the general equations (18) to (20), the problem of finding the optimal angles becomes one of specifying the weighting factor or the cost per unit length. Power losses do not represent the only important criterion involved in the operation of a stream channel. Because of its complexity, a river system may behave optimally with respect to several other factors, and it may not be possible to isolate all of them.

One can formulate other optimality criteria. For instance, the junction point may be located so as to minimize total flow resistance (R_t). From Leopold et al. (1964, p. 157), flow resistance per unit area on the boundary of the channel is

$$R = \tau/v^2 = \rho \mathrm{gSA}/(2\mathrm{D} + W)v^2, \tag{22}$$

where τ *is* shear stress on the channel boundary surface; v velocity; A, cross-sectional area; D, channel depth; and W, channel width. Given the length of the channel, total resistance to flow induced by the channel is

$$R_t = (\rho \mathrm{gSA}/v^2)l\,, \tag{23}$$

thus, yielding a cost per unit length

$$\omega = \rho \mathrm{gSA}/v^2. \tag{24}$$

Since cross-sectional area is the product of width and depth, then I have

$$\omega = \rho \mathrm{gSWD}/v^2.$$

(25)

Another possible way of defining the weights ω_i is to minimize the resisting force (F), which is

$$F = \rho \mathrm{gSA}l, \tag{26}$$

giving a weight

$$\omega = \rho \mathrm{gSWD}. \tag{27}$$

Finally, channel form parameters could also be used as a criterion. For example, costs of transport and channel maintenance increase with channel volume (V), given by

$$V = Al = WDl. \tag{28}$$

Hence, to minimize channel volume, the weight

$$\omega = WD \tag{29}$$

Is to be substituted into (18) to (20).

These optimality criteria are not exhaustive and others could be derived. It is important, however, to note that all of them could be simplified using the hydraulic geometry relationships (Leopold and Maddock 1953). By so doing, one can develop a general model of optimal angular geometry applicable to all criteria. Given that

$$W = aQ^b \tag{30}$$

$$D = cQ^f \tag{31}$$

$$v = iQ^m \tag{32}$$

$$S = tQ^z \tag{33}$$

and

$$b + f + m = 1.0 \tag{34}$$

and substituting these expressions into the weight equations (6), (24), (27), and (29), I get

$$\omega = \rho g t \; Q^{1+z}$$

(35)

for minimum power losses,

$$\omega = \rho g t a c \; Q^{1+z-m} \tag{37}$$

for minimum resisting force, and

$$\omega = \alpha c \; Q^{1-m} \tag{38}$$

for minimum volume. Therefore, the weights for all criteria have a similar mathematical form

$$\omega = jQ^{1+k}, \tag{39}$$

where j is a constant assuming fluid density is constant. Furthermore, ω is a function of the rates of change in slope and velocity with discharge? The weights for the receiving and tributary streams can be expressed in terms of the symmetry ratio α. Using equations (13) and (14) into (39) I have

$$\omega_0 = j\big(Q_1(1 + \alpha)\big)^{1+k} \tag{40}$$

$$\omega_1 = jQ_{1^{1+k}} \tag{41}$$

$$\omega_2 = j(\alpha Q_1)^{1+k} \tag{42}$$

These expressions can be substituted into (18) to (20), and, after simplification, the general optimal angular geometry model reduces to

$$\cos\theta_1 = \frac{(1+\alpha)^{2x}+1-\alpha^{2x}}{2(1+\alpha)^x} \tag{43}$$

$$\cos\theta_2 = \frac{(1+\alpha)^{2x}+\alpha^{2x}-1}{2a^x(1+\alpha)^x} \tag{44}$$

$$\cos\psi = \frac{(1+\alpha)^{2x}-1-\alpha^{2x}}{2a^x} \tag{45}$$

where

$$x = 1 + k \quad \text{and} \quad k < 0. \tag{46}$$

Hence, optimal angles are a function of two parameters: α,the symmetry ratio and k, computed from the hydraulic geometry relationships. The value of the exponent k, varies with the selected optimality criterion. .

Implications of the General Optimal Angular Geometry Models

Let us now assess the effects of the parameters on the optimal angles and discuss some implications of the models. The symmetry ratio greatly affects the angles of entry θ_1 and $\theta_{2.}$ This can be shown for a constant value of $k(k = -0.4)$ by plotting angles against the symmetry ratio (Fig. 4).

When two tributaries of the same size merge, θ_1 is equal to $\theta_{2.}$. Hence, symmetry in size implies angular symmetry. As a decreases, that is, one tributary becomes much larger than the other, $\theta_{2.}$ becomes wider than θ_1 When much larger than the minor tributary, the major tributary tends to lie in the prolongation of the receiving stream and θ_1 is very acute. The general optimal angular models have characteristics similar to the Hortonian model. Although angles of entry are greatly affected by $\alpha,$ the junction angle is nearly constant for all values of a. These trends, illustrated here for $k = -0.4,$ hold for all possible values of $k.$

Variations of it also affect the values of the optimal angles. The value of k must

Be smaller than 0 for optimal angles to exist. This is consistent with the fact that slope decreases with discharge and velocity increases with discharge. Hence, one expects for all optimality criteria that k is negative. As k takes lower values, the optimal junction angle increases. This relationship is illustrated in Figure 6 for two values of $\alpha.$ As seen earlier, junction angles are insensitive to variations in the symmetry ratio and the two curves in Figure 5 are nearly superimposed. The effect of the symmetry ratio on the optimal junction angles varies with the values of k considered. ψ Increases with a when k is larger than —0.5 and smaller than — 1.0. Otherwise, ψ decreases slightly with the symmetry ratio.

Because optimal junction angles become wider as k decreases, it implies from the optimality criteria that a sharp decrease in slope (low z) and/or a marked increase in velocity (high m) with discharge will be associated with wider junction angles. On the other hand, if cross-sectional area increases very fast with discharge, in a downstream direction, angles will tend to be acute. Hence, a small tributary (i.e., low discharge) will branch at a wide angle with a large tributary.

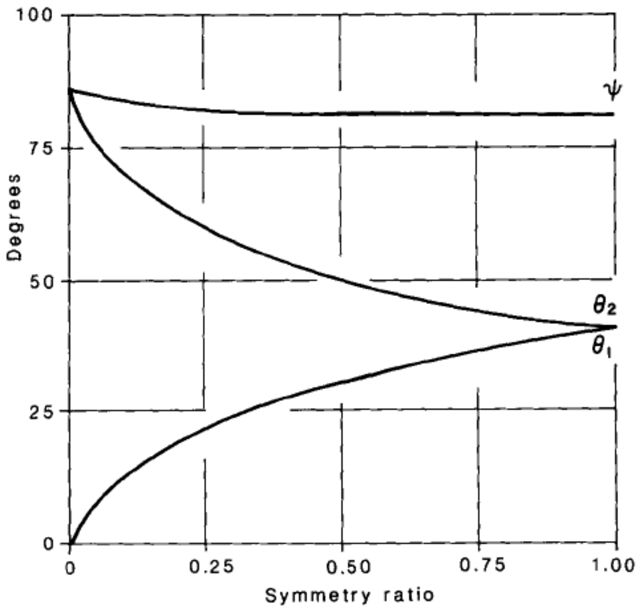

Figure. 4. Effect of the Symmetry Ratio on the Optimal Angles Assuming $k = -0.4$

CONCLUSION

Angular geometry models of river branching rest heavily on the optimality principle. In reviewing the existing models of river branching, I have proposed an alternative interpretation of the Hortonian model as minimum power losses per unit discharge model, and I have shown that Howard's minimum power model is only one particular application of

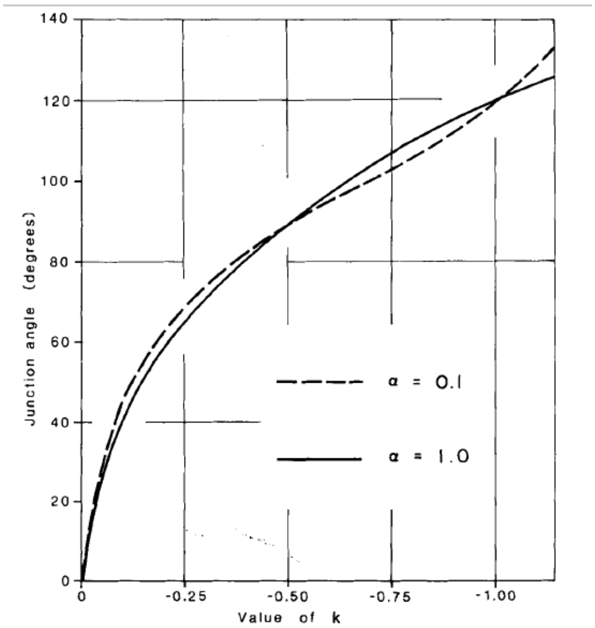

Figure. 5 Effect of the Value of k **on the Optimal Junction Angle for Symmetrical** $(\alpha = 1.0)$ **and Asymmetrical** $(a. = 01)$ **Branching**

the general optimization process derived by Zamir (1976). Indeed, models of the angular geometry of a river junction are derived from a minimization of the sum of weighted channel lengths. power losses, the criterion used by Howard (1971), is only one way of specifying the weights that represent some costs involved in the operation of the system.

By considering alternative optimality criteria such as minimum resistance to flow, minimum resisting force, and minimum volume, it is possible to generalize optimal angular geometry models. Optimal angles are a function of two parameters: the relative size of the tributaries and the hydraulic geometry relationships involved in the specification of the weights. Although the general model is flexible and provides a theoretical framework for future work, the problem of specifying the weights is still crucial. The criteria proposed in this paper only partially define the costs of operating a river system. None of them include the costs associated with sediment transport, channel roughness, and bed-forms. The complexity of a river system is obviously impossible to render with a single factor, and this could affect the applicability of the model.

Empirical work should help determine which criterion, if any, seems most appropriate to explain the angular geometry of rivers. Preliminary findings suggest that the general optimal model is a good predictor of the average angles in four typical fluvial networks from Devon, England (Roy 1982). For these systems, the minimum power losses criterion provided the best prediction. It is not inconceivable, however, that the best criterion varies with environmental conditions.

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