CALCULUS VARIATION OF BASALTIC RIVER BED PROFILES

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Figure:1 The Bainganga river of the Pachmarhi is originated from the trap high land

ABSTRACT
Appearance to the analysis of Basaltic river bed profiles of Pachmarhis and equilibrium longitudinal profiles and slopes are considered. The first one deals with a solution to the equation of Basaltic rock bed continuity in the equilibrium case and the second appearance deals with the application of variation principles. Both appearances considered and results obtained for the study of Basaltic longitudinal profiles of streams.

KEY WORDS Mathematical models, Basaltic rock beds, Calculus of variations.

The Drainage system of the Pachmarhis (Crookshank,H,1936, Dongre.N.L.1997,2013) is the most conspicuous one is the high Basalt land scape. All the rivers of the Pachmarhis are originated in the Trap high land, notably the Denwa, the Sonbhadra, the Bainganga, (Fig:1) the Dudhi, and the Tawa. The rivers are flows parallel to the strick of the hills. The rivers have cut deep channels through the highest hills(Fig:2). The rocks are the Deccan Trap Basalt which cap the hilltops on either side, are also found in the valley floor (Fig:3).To study the drainage systems of the Pachmarhis, the mathematical models are explored which can determine a stability and equilibrium state for the Basaltic river bed profile.(Fig:4) It is also necessary to have some information on the stability of the drainage systems over time. In this context some methods for the mathematical model of stable and equilibrium states of some widely known natural systems are given; namely for longitudinal river profiles and slope profiles. Two approaches for this can be used successfully, the first solving the continuity equation for sediment in a stationary case, the second obtained by applying variational principles (calculus of variation). Using the first approach, the continuity equation considered from geomorphological points of view. To describe the long term evolution of longitudinal river profiles, the continuity equation is traditionally written as follows:

$$\frac{\partial z}{\partial t}+\frac{\partial q}{\partial x}=0, q=k\left(-\frac{\partial z}{\partial x}\right)^{n} \tag{1}$$

Sediment discharge is also proportional to gradient $(n=1) q=-k\frac{\partial z}{\partial x}$ and equation (1) reduces to a diffusion equation. In the expression for sediment discharge a different exponent of gradient is used within the range $1 \leqslant n \leqslant 2$ The coefficient k can be expanded as: $k=k_1(x)f(Q)$, where $k_1(x)$ characterizes the change of physical properties of the sediments and the bed along the stream, and $f(Q)$ is the function of unit water discharge.

Most hydraulic dependence, as will be shown below, is likely to be reduced to a form suitable for the integration of equation (1):

$$G=\lambda_1\overline{Q^{m_i}}i^{n_i}, q=\lambda_2 Q^{m_2}i^{n_2} \tag{2}$$

where $\lambda_1,\lambda_2,m_1,m_2,n_1,n_2,=$ const, $\bar{Q}$ is total water discharge, $i=-\partial z/\partial x$ and G,q are respectively the total sediment discharge, and the sediment discharge per unit flow width (unit sediment discharge).

To make calculations for stable and equilibrium river bed and slope profiles, one valid procedure is to look for stationary solutions of equation (1) when $\partial z/\partial t=0.$ A more general case when $\partial z/\partial t=$const was considered in the work of Smith and Bretherton, (1972). It corresponds to the case of an equilibrium river bed and slope profiles in a region of tectonic movements

In a geomorphological sense the condition $\partial z/\partial t=0$ (height profile marks do not change in the course of time) may correspond to two possibilities:

I. a completely stable slope profile or channel, with sediment transport equal to $(q = 0)$, and

II. a dynamic equilibrium profile characterized by constant sediment discharge throughout $(q = \text{const.})$

Figure: 2, The Basaltic river bed of Denwa river

The second possibility is mainly observed in river systems of the Pachmarhis. Their most important regularity is the reduction of sediment transport capacity along a channel (the sediment discharge of the flow being in a saturated state). This is the dynamic equilibrium profile described. While analyzing river beds this profile will be borne in mind. If sediment discharge is written in the general form

$\left(i = \sin\phi = -\frac{\partial z}{\partial x} \Big/ \sqrt{\left(1 + \left(\frac{\partial z}{\partial x}\right)^2\right)} \right)$, then

$$q = k_1(x) f\big(Q(x)\big) \left[-\frac{\partial z}{\partial x} \Big/ \sqrt{\left(1 + \left(\frac{\partial z}{dx}\right)^2\right)} \right]^n \quad (3)$$

where n is the same as in equation (1). For $n = 1$ the analytical solution is then as follows

$$z = \pm C_1 \left[\int dx / \left(k_1^2(x) f^2\left(Q(x)\right) - C_1^2 \right)^{1/2} \right] + C_2 \tag{4}$$

where C_1 , $C_2$are the constants of integration·.

Thus, to obtain particular forms for the stable equilibrium bed and slope profile the aim is to define the functions $k_1(x), f\left(Q(x)\right)$ and the parameter n. The results have been obtained for the Pachmarhis model, in which, instead of the unit discharges, the total discharges are being considered. The proof of this procedure is clear from equation (7)

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[(ax + \bar{Q}_0)\frac{\mathrm{dz}}{\mathrm{dx}}\right] = 0, z(0) = H, z(l) = 0, \tag{5}$$

where $\overline{Q}_0 = \bar{Q}(0), G(0) = -k_1 \overline{Q_0}\,(dz/dx), k_1 = \text{Const}$.

The solution of equation (5) leads to a stable (equilibrium) concave logarithmic profile for the bed

$$z(x) = H \ln \left(\frac{ax+\bar{Q}_0}{al+\bar{Q}_0}\right) \Big/ \ln\left(\frac{\bar{Q}_0}{al+\bar{Q}_0}\right). \tag{6}$$

This concave logarithmic profile coordinates well with the calculated and measured (natural) equilibrium longitudinal profiles of ravines in rocks of the Pachmarhis region. Experiments in the modeling of rill erosion have also shown that after 30 minutes of the experiment the rills were stabilized, with a constant sediment discharge at all cross-sections of the rill expressed as:

$$G = M(\,\bar{Q}i)^{\delta}$$

where M =const; $1 \leqslant \delta \leqslant 1.5$ This expression coordinates well with the expression for a total sediment discharge in equation (5).

It is noted that the equilibrium longitudinal profiles, which are well approximated by a modified Bessel function, are equally well approximated by the logarithmic function (equation (6». The analysis shows that when the coefficient of the water discharge (a) increases, the gradient at the $x = 0$ creases while that at $x = 1$ diminishes towards zero: i.e. the profile becomes more and more concave. When Q_0 increases (for a = const) the profiles at the points $x = 0$, tend to the value $(-H/l)$: i.e. the profile becomes increasingly rectilinear.

The three works referred to above directly or indirectly confirm the relevance of model (5) in some cases. Next, the assumption of a linear increase in $\bar{Q}(x)$requires discussion. It is well known that total water discharge is proportional to the watershed area (A): $\bar{Q} = \sigma IA$, where σ is the coefficient of runoff, I is the rainfall intensity (this expression is formally known as the rational method and A, in the case of an approximately rectangular (watershed) configuration (e.g. a ravine), increases linearly by x. A direct connection between the length of small channels and their watershed area was obtained.

Consider now the continuity equation from a hydraulic point of view. In drawing up the sediment balance for an elementary volume (length Δx and bed width B), the equation after some simplification takes the form:

$$\frac{\partial G}{\partial x} + \frac{\partial (zB)}{\partial t} = 0 \quad (7)$$

where G is the total sediment discharge $(\mathrm{m}^3/\mathrm{S})$related to by the expression $G = Bq$.

Figure: 3, The Sonbhadra river have cut deep channel through the highest hills

Since B varies, equations (1) and (7) can be seen to differ. One additional factor to be considered is that with a large bed width variation (along the flow length) in equation (1), the cross-flow sediment transportation taken in to account, using the two-dimensional continuity equation $\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial z}{\partial t} = 0$. A prospective model can be made by the closure of equation (7), using equations of the forms:

$$G = \psi_1(\overline{Q}, v, h, d) \quad (8)$$

$$Q = Bhw = \phi_1(x) \quad (9)$$

$$B = \psi_2(\overline{Q}, d, -\partial z/\partial x) \quad (10)$$

$$-\frac{\partial(z+h)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{v^2}{2g}\right) + \frac{1}{g}\frac{\partial v}{\partial t} + \frac{v^2}{C^2 h} \quad (11)$$

$$d = \varepsilon(x) \quad (12)$$

$$C = \eta(x, d, h, -\partial z/\partial x) \quad (13)$$

where $\bar{Q}$ is in m^3/s, and d is the average diameter. Theoretical and experimental studies led to the following function for total sediment discharge in rivers with a sand bottom

$$G = k_2 Q v^2 \quad (14)$$

Calculating a stable Basaltic River bed profile using the formula of Chezy-Manning instead of equation (11)

$$v = \frac{1}{N} h^{2/3} \sqrt{i} \quad (15)$$

and solving the set of equations (9), (14) and (15) together I come to the following expression for G:

$$G = k_s N^{-6/5}. B\left(\bar{Q}/B\right)^{9/5} i^{3/5}$$

Choosing the function (10) for Basaltic beds.

$$B = A_1(\bar{Q})^{0.5} i^{-0.2} \quad (16)$$

I get the expression

$$G = k_s n^{-6/5} A_1^{-4/5} (\bar{Q})^{7/5}\, i^{19/25} \quad (17)$$

and equation (7) for $\partial z/\partial t = 0$ (recalling that $i = -\partial z/\partial x)$ leads to a concave profile (when $\bar{Q}\,(x) = ax + \bar{Q}_0$). Equation (2) brings together all the formulae for sediment discharge having the structure $G = Dv^p h^s \bar{Q}$. Thus the solution for stable calculation is reduced to one of defining $\bar{Q}\ (x) = \phi_1(x)$, which is easily done with the help of reference books on average long-term water discharge.

The approaches which are used while designing equilibrium fluvial system is a model based on the continuity equation for sediment in a stationary case can be used for forecasting the non-reversible deformation of longitudinal river profiles. Here is a query arises as to what the sediment transport capacity will be in order not to disturb the maintenance of an equilibrium bed profile. The basis of the problem and the approach to its solution are as follows. Let there be an equilibrium channel bed profile which is described with help of the continuity equation in the equilibrium case. The calculated equilibrium river bed profiles before and after discharge can be obtained with the final step of the solution to determine sediment discharge.

To deal with the approaches given above, modelling of equilibrium fluvial systems allow in areas of tectonic movement and a set of equations is depicted describing the equilibrium profile for each river segment. A necessary condition arises that continuity of elevation is maintained between at tributary junctions.

Let us now go through the execution of variational principles to longitudinal equilibrium river bed and river profile calculations.

First of all the main general statements of calculus of variations are briefly reviewed. The calculus of variations (variational principles) is broadly presented in many fields of science. (Courant and Hilbert, 1953; Gelfand and Fomin, 1963; Becker, 1964). The calculus of variations is basically concerned with changes in functional.

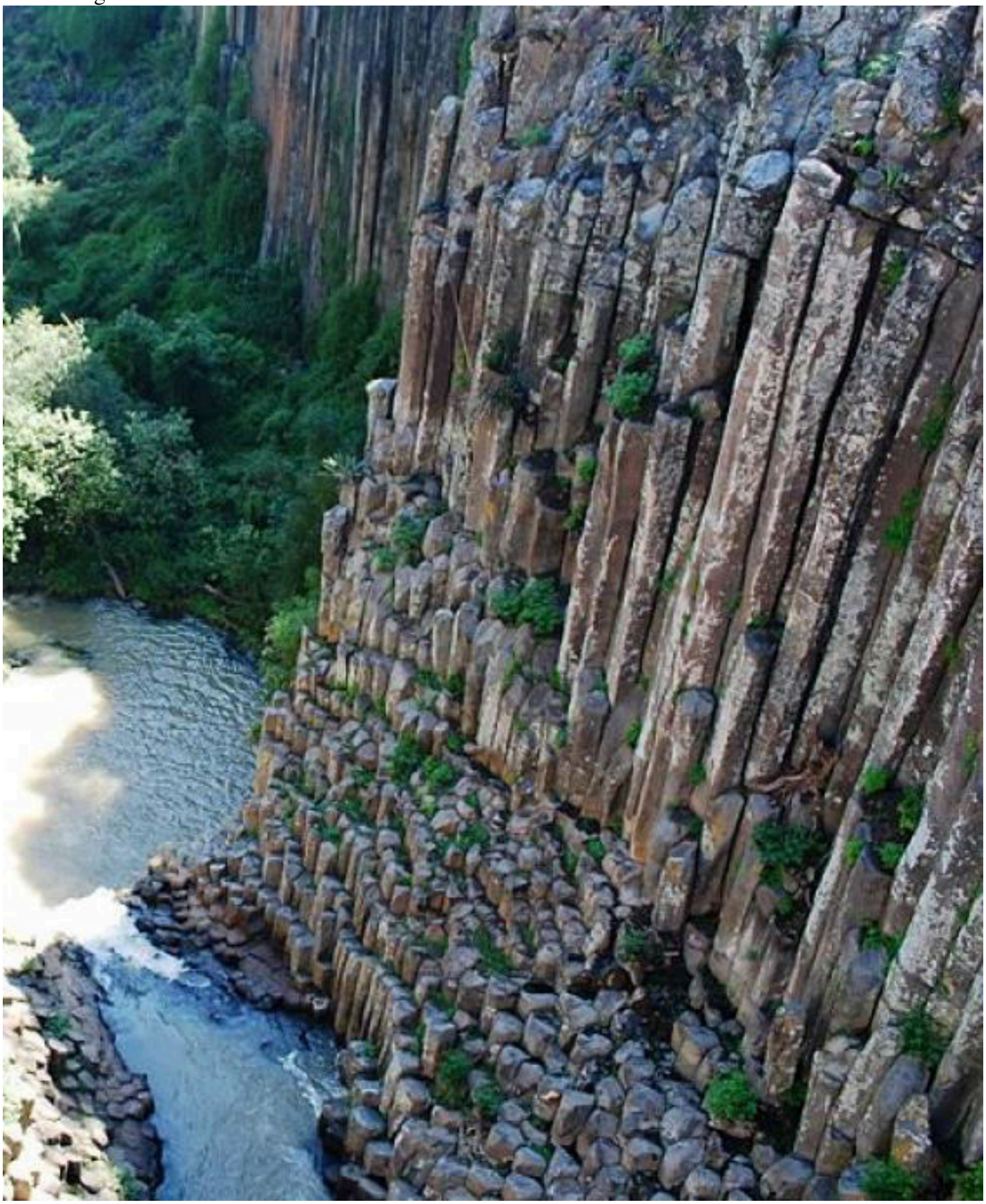

Figure: 4, Columnar joints and Basaltic River bed of the Dudhi

A functional is a correspondence between a function in some class and the set of real numbers, for example $\Phi(z) = \int_a^b F(x, z, z')\,\mathrm{d}x, z' = \frac{\mathrm{dz}}{\mathrm{dx}}$. For the functional given above the space of admissible functions might be those functions which are continuous and have continuous first derivatives on the interval *a* to *b.* In the calculus of variations a function is stationary when its derivative vanishes. Similar ideas are applicable to functionals. These ideas leading to the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial F}{\partial Z\prime}\right) - \frac{\partial F}{\partial z} = 0 \qquad (18)$$

which represents a necessary condition for $\Phi(Z)$ to be stationary (necessary condition for extremum). The functions which satisfy equation (18) are called extremals. Whether the extremal makes the functional is not easily answered definitely. Note, that two boundary conditions need be satisfied for equation (18). Thus I are led to the problem of an unconventional extremum.

Variations can be dealt with subject to constraints and these are easily handled using Lagrangian multipliers. The problem is to make the functional Φ stationary subject to the condition that the functional has a prescribed value P_1:

$$\Phi = \int_a^b F(x, z, z')\,\mathrm{d}x : P = \int_a^b L(x, z, z')\mathrm{d}x \qquad (19)$$

To derive the Euler-Lagrange equation for a constrained problem (problem for a conventional extremum) the Euler-Lagrange equation is derived for the integrand $F^* = F + \lambda L$:

$$\frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{\partial F}{\partial z\prime} + \lambda \frac{\partial L}{\partial z\prime}\right) - \frac{\partial}{\partial z}(F + \lambda L) = 0 \qquad (20)$$

The solution to equation (20) has two undetermined constants plus the unknown parameter λ (Lagrange multiplier). These are determined by the two boundary conditions and $P = P_1$ Thus, the problem for a conventional extremum is equivalent to extremization $\Phi + \lambda P$ functional.

Consider now the application of variational principles to the determination of equilibrium and stable states for the fluvial and slope systems. Some general ideas are also on this problem. It may be noted that an equilibrium river bed profile are in a state of minimum energy dissipation. Using this technique while analysing longitudinal profiles of rivers in the form of $\int \gamma\,\bar{Q} i dx$ →Langbein and Leopold (1964) arrived at the relationship $i \sim (\bar{Q})^{-1}$. It was arrived at as an equilibrium solution of the continuity equation $\partial z / \partial t = 0$ for sediment discharge G set by equation (2) with $m_{1,} \eta_1 = 1$.

Consider the variational principle applied to the design of slopes which are subjected to minimum erosion. The erosion at every point of a slope is known to be a function of gradient and distance from a divide (along a horizon position) (Carson and Kirkby, 1972)

$$R = F\left(-\frac{\partial z}{\partial x}, x\right) \qquad (21)$$

In this case the erosion from a whole slope *(W)* is expressed with the integral (functional) (due to a small inclination of slopes $\sin\varphi \approx \tan\varphi = -\frac{\partial z}{\partial x}$ is supposed)

$$\Phi(z) = W = \int_0^1 F\left(-\frac{\partial z}{\partial x}, x\right)\mathrm{d}x \qquad (22)$$

To extremize the integral (functional) the variation principle are used. The problem is solved for a conventional extremum by equation (20) when the square (material's volume for a space problem) under a slope profile is set by a constant value and also for an unconventional extremum (by equation (18)

Equation (21) will in the form (Carson and Kirkby, 1972) $R = cx^m\,(-\partial y/\partial x)^n$·.Typical values of parameters *m* and *n* lie in the ranges *m* = (0'4; 0'5), *n* = (0'5; 2) . The solution of the problem for a conventional extremum

$$\Phi(z) + \lambda P(z) = \int_0^l \left[cx^m\left(-\frac{\partial z}{\partial x}\right)^n + \lambda z\right]dx \to \text{exte}; P(z) = \int_0^l z\mathrm{d}x = P_1 = \text{const};$$

$$z(0) = H; z(l) = 0; \lambda - \text{Lagrange multiplier} \qquad (23)$$

may be obtained in the form:

$$z = -\int (cn)^{1/1-\eta}\, x^{m/1-\eta}\,(\lambda x + C_1)^{1/\eta-1}\, ax + C_2 \qquad (24)$$

where $C_{1,}C_2$ are integration constants.

Taking the parameter values $n = 1/2, n =$ I obtain a concavo-convex profile from equation (24)

$$z_{\text{extr}(x)} = \frac{\lambda}{3c}x^{1/2}(l-x) + H\left(1 - \sqrt{\frac{x}{l}}\right); \lambda = 15c\left(\frac{3P_1 - Hl}{4l^{5/2}}\right). \qquad (25)$$

The inflexion is at distance $x = (8l - 5H)/15$.The analysis shows the expression (25) to characterize a minimum wash:

$$W_{\text{extr}} = W_{\text{min}} = \frac{1}{4}C\left[2H^2l^{-1/2} + \frac{5}{2}l^{-5/2}(3P_1 + Hl)^2\right] \qquad (26)$$

For instance, when,$P_1 = Hl/2$, $W_{\text{min}} = (21/32)\, cH^2l^{-1/2,}$ and for wash along a rectilinear slope profile $z(x) = H(1 - x/c)$ (for which the square under a slope profile also equals $P_1 = Hl/2$) is equal to $W_{\text{rect}} = 2/3\; cH^2l^{-1/2}$. Thus, when $m = 1/2, n = 2, P_1 = Hl/2$wash along a rectilinear slope profile is nearly minimum ($W_{\text{rect}} \approx W_{\text{min}}$).

The solution of the problem for an unconventional extremum

$$\Phi(z) = W = \int_0^l cx^m\left(-\frac{\partial z}{\partial x}\right)^{\text{n}} \text{dx} \to \text{extr}; \text{z}(0) = H, z(l) = 0 \qquad (27)$$

can be obtained in the form

$$z_{\text{extr}}(x) = -(C_1/c_n)^{\frac{1}{n}-1}\left[x\frac{m}{1-m} + 1\Big/\left(\frac{m}{1-n} + 1\right)\right] + C_2 \qquad (28)$$

For instance, $m = \frac{1}{2}, n = \frac{1}{2}$ (and $z(0) = H, z(l) = 0$), I obtain a convex profile which causes maximum erosion: $z = H\,\left(1 - \frac{x^2}{l^2}\right)$.The minimum for the set of functions (Figure: 5)

$$z = H\,\left[1 - \left(\frac{x}{l}\right)^{\alpha}\right] \qquad (29)$$

is obtained when $\alpha \to 0, i.e$ the concave profile turns into a profile in the form of a cliff edge. Thus, for slope profiles subjected to minimum erosion I have an infinite derivative at the point $x = 0$, which does not meet the assumption of small angle. Thus, in some cases sin φin formula (21) should be considered instead of tan π, as in equation (3). In these cases the problem for an unconventional extremum is set in the form

$$W = C \int_0^l x^m \left[-\frac{\partial z}{\partial x} \Big/ \sqrt{\left(1 + \frac{\partial z}{\partial x}\right)^2} \right]^n \mathrm{dx} \to \mathrm{extr}; \mathrm{z}(0) = H, z(l) = 0 \quad (30)$$

The first integral of Euler-Lagrange equation for this problem was obtained as follows

$$x^m \left(-\frac{\mathrm{dz}}{\mathrm{dx}}\right)^{n-1} = C_1 \left[1 + \left(\frac{\mathrm{dz}}{\mathrm{dx}}\right)^2\right]^{1+\frac{n}{2}} \quad (31)$$

Only when $n = 1$ an analytical solution of equation (31) can be obtained which is reduced to integral taking $z(x) = \int \sqrt{\left(Gx^{2m/3} - 1\right)} \mathrm{d}x$. in other cases the problem is difficult to solve.

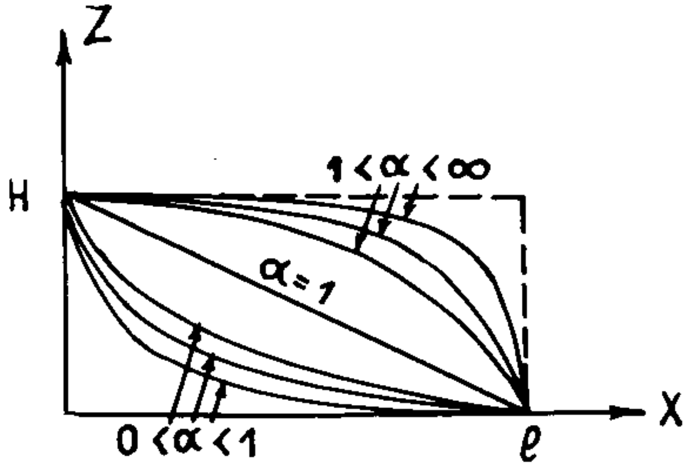

Figure 5. Set of parabolas

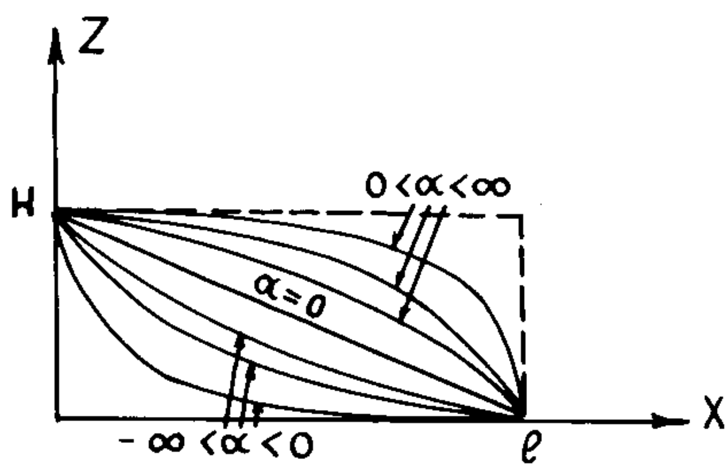

Figure 6. Set of exponential functions

Another approach to the problem solution of equation (30) can include the finding of functional extremum Won definite sets of curves. It is then possible to take a set of functions (29) or: $z = H\left[\frac{\exp(\alpha(x/l))-\exp\alpha}{1-\exp\alpha}\right]$ (Figure 6). The last set allows us to take a wider range of profiles. The derivative at the point $x = 0$, continuously varying in the intervals $(-H/l, 0)$ and $(-\infty, -H/l)$ with a variation of a in the intervals$(0, \infty), (-\infty, 0)$. In the former case I have the set of convex curves, and in the latter I have the set of concave curves. Here the rectilinear profile corresponds to $\alpha = 0$ (Figure 6), which follows from the Taylor series expansion, at small α: $exp(\alpha(x/l)) \approx 1 - \alpha\left(\frac{x}{l}\right)$; $\exp\alpha \approx 1 - \alpha$.

The problem (30) with this set of curves is reduced to one of minimizing, with respect to x the integral

$$W_n^m = C\int_0^t \frac{x^m \psi^n(x,\alpha)dx}{[1+\psi^2(x,\alpha)]^{n/2}}$$

$$\psi(x,a) = \frac{H\alpha \exp\left(\frac{\alpha}{l}x\right)}{l(\exp\alpha - 1)}$$

Where

The approaches discussed here and the results obtained can be applied in practice to hydraulic engineering works and to the design of equilibrium and stable river beds and slope profiles.

List of notations

$z -$ height, $L, H = z(x = 0)$
$l -$length of slope or river bed, L
$x -$ space coordinate (distance from the divide), L
$t -$time, T
Q—unit water discharge (water discharge per unit flow width), L^2/T
$\overline{Q}$ — total water discharge, L^3 / T
q — unit sediment discharge, L^2 / T
G —total sediment discharge, , L^3 / T
i —gradient
ϕ —slope angle
B —bed width, L
h —average depth, L
v —mean velocity, $\frac{L}{T}$
N —Manning resistance coefficient
d —average sediment diameter, L
g —gravitational acceleration, L/T^2
W —erosion (wash) from a whole slope, $L^3/L = L^2$
R —erosion at every point of a slope, L
m, n —exponents of $R, R = cm^m i^n$(n —also exponent of$q, q \sim i^n$)
λ —Lagrange multiplier
c —coefficient in R
C_1, C_2 —integration constants
σ —coefficient of runoff

I −rainfall intensity, L/T
A −watershed area, L_2
C − Chezy coefficient, $L^{1/2}/T$

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